A Behavioral Model for Participation Games with Negative Feedback*

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Abstract

We study participation games with negative feedback, i.e. games where players choose either to participate in a certain project or not and where the payoff for participating decreases in the number of participating players. We use the replicator dynamics to model the competition between different behavioral rules that specify how to play the game in a repeated setting. This results in an analytically tractable model which is able to describe the type of behavior found in the experimental and computational literature. We find that an increase in the number of players destabilizes the unique symmetric mixed strategy Nash equilibrium. The time series of perpetually fluctuating participation rates typically exhibits linear autocorrelation structure and underparticipation. We investigate whether this time series structure can be exploited, and we relate underparticipation to the payoff structure of the participation game.

Keywords: Participation games; Evolutionary game theory; Nonlinear dynamics.

JEL Classification: C72; C73.

1 Introduction

Many decisions, such as companies choosing whether or not to enter a new market or to invest in a new technology, commuters choosing a route to the workplace, workers deciding on union membership or citizens deciding to vote or not, involve the choice between only two alternatives. These situations can be modeled as participation games where each player has to decide whether or not to participate in a certain

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'project' and where the payoff associated with participating depends upon the number of other players participating. Where different players compete for a scarce resource, as in market entry or route choice, these payoffs decrease as the number of participating players increases. In this case there is a *negative feedback* from the aggregate participation rate to individual payoffs. Public good provision, union membership or technology adoption, on the other hand, may be characterised by *positive feedback*, where payoffs for participating increase with the number of participating agents. The character of the feedback is particularly relevant when participation games are played repeatedly. In a negative feedback participation game, when a player believes that few players will participate in the next period, he\she is more likely to participate. However, if many other players think this way, the participation rate in the next period will in fact be high. In contrast, for positive feedback participation games beliefs about participation rates are self-reinforcing.[1]

In this paper we focus on negative feedback participation games, where players have to compete for a scarce resource. Typically, these participation games allow for multiple Nash equilibria. When communication is impossible, and in the absence of market institutions that act as a coordination device, players have to rely on their private beliefs and observations of previous aggregate outcomes to solve this non-trivial coordination problem. Negative feedback participation games can be divided in two classes of games, which we refer to as *market entry games* and *route choice games*, respectively. In the first the payoff for participating depends upon aggregate decisions of other players, whereas the payoff for not participating is constant. This is for example the case when firms have to decide to enter a new market or not. In route choice games, on the other hand, both alternatives, participating and not participating, are subject to strategic uncertainty in the sense that their respective payoffs depend on the decisions made by the other agents. A typical example are commuters traveling every day between their residential area and their office facilities and having to decide which of two possible roads they should take.

Market entry games, with payoffs for entering the market linearly decreasing in the number of entrants, have been extensively analysed experimentally, see e.g. Sundali *et al.* (1995), Erev and Rapoport (1998), Rapoport *et al.* (1998), Zwick and Rapoport (2002) and Duffy and Hopkins (2005). A robust finding from this literature is that participation rates keep on fluctuating and that players do not coordinate on a pure strategy Nash equilibrium. Aggregate behavior seems to be roughly consistent with the symmetric mixed strategy Nash-equilibrium. At the individual level, however, subjects typically do not randomize and a large variation of strategies can be observed, with some subjects always participating, others never participating and yet other subjects conditioning their behavior on the outcome in previous rounds. These results are confirmed in experiments of the route choice type. Meyer *et al.* (1992), for example, consider suppliers who have to choose between two locations to sell their product and Iida *et al.* (1992) and Selten *et al.* (2006) report on laboratory experi-

[1] In a related expectations feedback framework Heemeijer *et al.* (2006) present experimental evidence that shows there indeed exist substantial differences between behavior in environments characterised by positive expectations feedback and environments characterised by negative expectations feedback.

ments where subjects have to choose between two roads for a number of consecutive periods. There is thus abundant experimental evidence that, particularly at the individual level, Nash equilibrium is not a good description of actual behavior, and that participation rates typically do not settle down. Goeree and Holt (2005) argue that a quantal response equilibrium is more succesful than the Nash-equilibrium in explaining data from market entry experiments. The reinforcement learning model (Erev and Roth, 1998) has also been advanced to explain the experimental results (see e.g. Erev and Rapoport, 1998 and Duffy and Hopkins, 2005). An important drawback of these explanations is that they assume that subjects randomize their participation decision, which is not supported by the experimental evidence.

The computational economics literature on negative feedback participation games starts out with Arthur (1994) who considers N players that independently and repeatedly decide whether to go to a bar or not. The night in the bar is pleasant (unpleasant) when N_c people or less (more than N_c people) go, whereas staying at home would result in an intermediate neutral experience. This so-called *El Farol* bar problem is, in fact, a market entry game with payoffs given by a step function. Arthur (1994) uses computer simulations to analyse the interaction of 100 agents each choosing from their own set of predictors for the participation rate. Agents make their participation decision on the basis of the selected predictor. Forecasting accuracy of the predictors determines which of them is chosen by the player.[2] The simulations show that fluctuations persist in the long run, although average participation seems to converge to the capacity of the bar. Moreover, regularities in participation rates are 'arbitraged' away by rules that predict cycles. Zambrano (2004) shows that the average participation rate coincides with the set of mixed strategy Nash equilibria of the prediction game that underlies the *El Farol* bar problem. Franke (2002) applies a reinforcement learning model to the *El Farol* bar problem and finds that the long run distribution of the probability to participate is either centered around the symmetric mixed strategy Nash equilibrium or is binomial with peaks at very low and very high probabilities to participate. He concludes that the long run outcomes are rather sensitive to the model specification. A 'route choice' variant of the *El Farol* bar problem is the so-called minority game. This game, introduced and studied by physicists (see e.g. Challet and Zhang, 1997 and 1998) using tools from statistical mechanics, has an odd number of players and positive payoffs only for the players making the minority choice. Bottazzi and Devetag (2003) present experimental results on a minority game with 5 players. A drawback of all of these computational models is that they are quite complicated and therefore typically analyzed by simulation methods only.

In this paper we introduce a framework where agents play the negative feedback participation game by using simple deterministic rules. Each behavioral rule prescribes exactly, given past aggregate outcomes, when to participate. An evolutionary competition between those rules, based upon the well-known replicator equation, determines the fraction of the population using each rule. These fractions depend positively upon the payoffs generated in the previous periods by the corresponding

[2]Notice that, although payoffs of the *El Farol* bar game are of the 'market entry'–type, by rewarding rules according to prediction accuracy the game is in fact transformed into a route choice game. We will return to this issue in Section 6.

rule. This approach has been applied in other environments (see e.g. Brock and Hommes, 1997, 1998, and Droste *et al.*, 2002) but is new for participation games. We show that our simple and tractable analytic model is able to describe the results of the high dimensional computational models and is consistent with the experimental evidence. Moreover, we establish that as the number of players (which we refer to as the *size* of the game) increases the Nash equilibrium destabilizes. Note that if players would play the symmetric MSNE fluctuations in the participation rate decrease as the number of players increases. The effect of the size of the game is of interest since laboratory experiments can typically deal only with up to 25 players whereas natural experiments such as the route choice problem or internet congestion involve many more players. We also show that the introduction of new rules which try to exploit any regularities in the time series of participation rates do not stabilize participation, but instead make the time series more irregular and more unpredictable.

The remainder of the paper is structured as follows. In Section 2 the negative feedback participation game is introduced, its Nash-equilibria are characterised and some illustrative experimental results are discussed. Section 3 introduces a framework for studying evolutionary competition between different behavioral rules and in Section 4 this framework is applied to the simplest possible environment. The relation between stability and the number of players is also discussed there. In Section 5 we investigate whether new rules can profit from regularities in the time series of participation rates and Section 6 relates the 'participation premium' found in experiments and in the numerical simulations to the difference between market entry and route choice games. Section 7 concludes. The appendix contains proofs of the main results.

2 A participation game with negative feedback

Consider a *market entry* game with N players. Each player chooses an action $a \in \{0,1\}$, where $a = 1$ stands for participating and $a = 0$ for not participating. The action space is given by $A = \{0,1\}^N$ and an action profile by $\mathbf{a} \in A$. By $\mathbf{a}_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_N)$ we denote the set of actions played by all players but player i. Payoffs $\pi_i(a_i, \mathbf{a}_{-i}; N_c, N)$ are given as:

$$\begin{aligned} \pi_i(0, \mathbf{a}_{-i}; N_c, N) &= \alpha, \\ \pi_i(1, \mathbf{a}_{-i}; N_c, N) &= \begin{cases} \alpha + \beta - \gamma & \text{if } \sum_{j=1, j\neq i}^{N} a_j < N_c \\ \alpha - \gamma & \text{if } \sum_{j=1, j\neq i}^{N} a_j \geq N_c \end{cases}, \end{aligned} \tag{1}$$

where the parameter N_c denotes the capacity of the project. Participating gives payoff $\alpha - \gamma$ if N_c or more of the other $N-1$ players participate, and payoff $\alpha + \beta - \gamma$ if less than N_c of the other $N-1$ players participate.[3] We assume $\alpha > \gamma$ to ensure that payoffs are always strictly positive and $\beta > \gamma$ to ensure that $a_i = 1$ is not a dominated strategy. The payoff α corresponds to some outside payment, γ to the cost or effort of participation and β to the (uncertain) return of a succesful project.

[3]Note that for $N = 2$ and $N_c = 1$, this payoff structure is similar to the well-known Hawk-Dove game (see e.g. Fudenberg and Tirole, 1991, pp. 18-19).

Notice that in many experimental market entry games the payoff for participating is linearly decreasing in the number of other entrants. The present formulation, using a step function for the payoffs, is equivalent with the payoff function for the *El Farol* bar game used in Arthur (1994) and Franke (2002). We chose this stylized version of the market entry game since it facilitates analyzing the effect of N on the dynamics. A strategy s_i for player i is the probability with which he \she chooses action $a = 1$. The strategy space is therefore given by $S = [0,1]^N$, $\mathbf{s} \in S$ denotes a strategy profile and $\mathbf{s}_{-i}$ the set of strategies for all players, except player i. We assume all players are risk neutral and want to maximize their expected payoffs. The expected payoff of playing a mixed strategy s_i is given by

$$
\begin{aligned}
\pi_i\left(s_i, \mathbf{s}_{-i}; N_c, N\right) &= \left(1 - s_i\right)\alpha + s_i\left(\alpha + \beta \Pr\left\{\mathbf{N}_{-i} \le N_c - 1\right\} - \gamma\right) \\
&= \alpha + s_i\left(\beta \Pr\left\{\mathbf{N}_{-i} \le N_c - 1\right\} - \gamma\right) \qquad (2)
\end{aligned}
$$

where $\Pr\left\{\mathbf{N}_{-i} \le N_c - 1\right\}$ is the probability that the number of other agents participating, $\mathbf{N}_{-i}$, which is a random variable, is strictly smaller than N_c. Obviously, this probability depends upon the strategy profile $\mathbf{s}_{-i}$.

2.1 Nash equilibria

The game described above has many pure strategy Nash equilibria (henceforth PSNE). Any strategy profile $\mathbf{s}$ such that exactly N_c players participate with certainty ($s = 1$) and the other $N - N_c$ participants abstain with certainty ($s = 0$) corresponds to a strict PSNE. Evidently, there are $\binom{N}{N_c}$ of these PSNE. Note that such a PSNE leads to an uneven distribution of payoffs, with exactly N_c players obtaining $\alpha + \beta - \gamma$ and the other $N - N_c$ players receiving α.
Now consider mixed strategy Nash equilibria (henceforth MSNE), where some players randomize between the two possible actions. We will establish that there exists a unique symmetric mixed strategy Nash equilibrium $s^* \in (0,1)$. If each player participates with probability s^*, the probability that the number of players participating is strictly smaller than N_c is given by:

$$
p\left(s^*; N_c, N\right) = \sum_{k=0}^{N_c - 1} \binom{N-1}{k} \left(s^*\right)^k \left(1 - s^*\right)^{N-1-k}. \qquad (3)
$$

Notice that $p\left(s^*; N_c, N\right)$ is a polynomial of degree $N - 1$ in s^*. In particular, $p\left(s^*; N_c, N\right)$ is the cumulative distribution function evaluated at $N_c - 1$ of a binomial distribution with $N - 1$ degrees of freedom and probability s^*. By the definition of a Nash equilibrium, s^* is a best response for player i only when, given that every player uses strategy s^*, player i is indifferent between participating and not participating. That is, at s^* we must have:

$$
\pi_i\left(1, s^*; N_c, N\right) = \alpha + \left(\beta p\left(s^*; N_c, N\right) - \gamma\right) = \alpha = \pi_i\left(0, s^*; N_c, N\right).
$$

Hence the equilibrium value of s^* is implicitly given as the solution to the following equation:

$$
p\left(s^*; N_c, N\right) = \frac{\gamma}{\beta}. \qquad (4)
$$

Since we are interested in the case where N becomes large, but where the (relative) capacity of the project remains the same, we define $b \equiv \frac{N_c}{N}$ and consider different values of N but constant values of b. The following proposition summarizes the properties of the symmetric MSNE.

Proposition 1 *For any N, $N_c < N$, α, γ, $\beta > \gamma$, there exists a unique symmetric MSNE s^* of the N-player participation game with payoff function (1). The value of s^* solves (4) and does not depend upon α. Moreover, $s^* \to b$ as $N \to \infty$ for all γ and β, and $s^* = b$ for all N when $b = 1/2$ and $\gamma/\beta = 1/2$.*

The exact value of s^* depends on the threshold value b and on the ratio γ/β. Generically it is unequal to b, but it approaches b as the number of players becomes large. This is illustrated by Fig. 1 which shows s^* as a function of N for $b = \frac{1}{2}$ and different values of γ/β. Furthermore $s^* = \frac{1}{2}$ for all N when $b = \frac{1}{2}$ and $\beta = 2\gamma$, that is when the standard payoff α is exactly in between the payoffs of a successful and a non-successful participation. We will use this specification as a benchmark for most of the numerical analyses in the rest of this paper. Note that when the symmetric MSNE is played, the total number of participating players $\mathbf{N}$ has mean Ns^* and variance $Ns^*(1-s^*)$. The participation rate sequence $\{x_t\}$, with $x_t = \frac{\mathbf{N}_t}{N}$, would therefore be randomly distributed around s^* with variance $\frac{s^*(1-s^*)}{N}$ and would have zero autocorrelations at all lags. Observe that as N becomes large, the distribution of x will converge to a point mass at s^*.

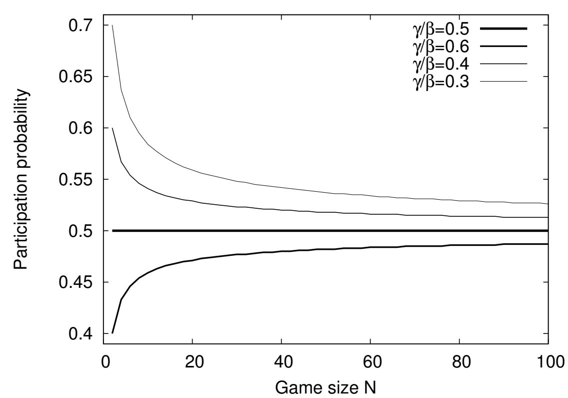

Figure 1: **Symmetric mixed strategy Nash equilibrium.** Every curve shows an approximation of the MSNE s^*, for N up to 100 and $b = \frac{N_c}{N} = \frac{1}{2}$. Each curve corresponds to a different value of γ/β. From top to bottom: $\frac{\gamma}{\beta} = \frac{3}{10}$, $\frac{\gamma}{\beta} = \frac{2}{5}$, $\frac{\gamma}{\beta} = \frac{1}{2}$ and $\frac{\gamma}{\beta} = \frac{3}{5}$.

Asymmetric MSNE, with players randomizing with different probabilities, also exist, for example, with $M < N_c$ agents always participating and the other $N - M$ agents randomizing with equal probability. In fact, in this case the players randomizing are playing the symmetric MSNE of the participation game with payoff structure (1) but with size $N - M$ and capacity $N_c - M$.

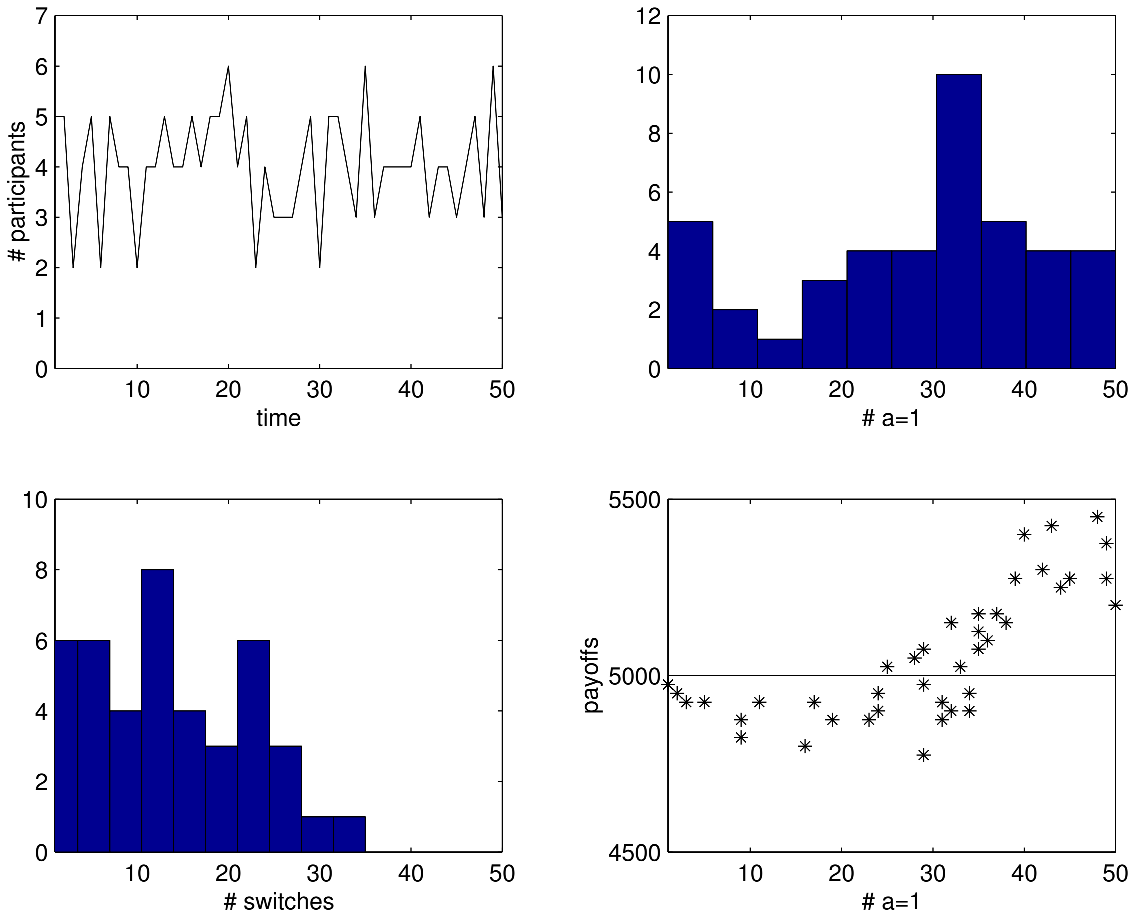

Figure 2: **Upper left panel**: Time series of number of participating players in experimental group 1. **Upper right panel**: Histogram of the number of particpations for the 42 participants. **Lower left panel**: Histogram of the number of switches. **Lower right panel**: Relation between individual payoffs and number of participations.

The PSNE has the characteristic that it extracts all rents from the game, but distributes them asymmetrically over the players, whereas the symmetric MSNE gives the same expected payoff to each of the players, but may lead to allocative inefficiencies. The symmetric MSNE is the only symmetric equilibrium. Moreover, it is also the unique evolutionary stable strategy (see Dindo, 2006).

2.2 Some experimental results

We will now briefly describe some experimental results on a negative feedback participation game from Heemeijer (2006). Our aim is not to provide a full-fledged analysis of the experimental data, but to illustrate and motivate the model introduced in this paper. The experiment was conducted in October 2005 and February 2006 at the CREED laboratory of the University of Amsterdam. It involved 6 groups of

$N = 7$ players, which had to make a participation decision for 50 subsequent periods. Group composition remained the same over the course of the experiment. Capacity was equal to $N_c = 4$ and payoff parameters were given by $\alpha = 100$, $\beta = 50$ and $\gamma = 25$. Moreover, a stochastic term ε_t from a symmetric triangular distribution on $[-25, 25]$ was added to the payoff for participating in every period. Subjects did know the value of α, but not those of β, γ and ε_t. Following the analysis above we have $\binom{7}{4} = 35$ PSNE. The symmetric MSNE s^* is implicitly given by $p(s^*; 4, 7) = \frac{1}{2}$. This gives $s^* \approx 0.5786$, which is slightly larger than $b = \frac{N_c}{N} = \frac{4}{7} \approx 0.5714$.
The results from this experiment are consistent with the experimental evidence discussed in the introduction. The upper left panel of Fig. 2 shows, for the first experimental group, the dynamics of the number of participating subjects over 50 periods. Behavior in the other groups was similar. Aggregate participation decisions are unstable and keep on fluctuating until the end of the experiment. Subjects did not coordinate on one of the PSNE and, at the aggregate level, the symmetric MSNE seems to provide a better description of the data. The first row of Table 1 shows that average participation rates in all groups are quite close to $s^* \approx 0.5786$, although there seems to be some 'underparticipation': for five of the six groups the average participation rate is somewhat lower than predicted by the symmetric MSNE.

	gr. 1	gr. 2	gr. 3	gr. 4	gr. 5	gr. 6	mean
$\frac{1}{50}\sum_{t=1}^{50} x_t$	0.5743	0.5943	0.5600	0.5629	0.5714	0.5543	0.5695
group-switches	25	14	25	25	26	22	$22\frac{5}{6}$
individual switches	$15\frac{5}{7}$	$8\frac{2}{7}$	$15\frac{4}{7}$	13	$17\frac{3}{7}$	14	14
% naivety	0.7364	0.7414	0.7156	0.8242	0.6885	0.7143	0.7330

Table 1: **Experimental results.** The first row gives the average participation rate for each group. The second row gives the number of times, per group, that participation changed from four or less to five or more subjects or vice versa. The third row gives, per group, the number of individual switches between participating and not participating, averaged over subjects. The last row gives the percentage of individual switches that follow directly after a negative payoff experience.

At the individual level, however, the symmetric MSNE is not supported by the data. The upper right panel of Figure 2, for example, shows a histogram of the number of times the 42 subjects participated. Apparently, some subjects participate almost always, whereas others participate almost never. This suggests that many subjects do not randomize. Further evidence is given by the second and third row of Table 1, which show that, although the number of switches in aggregate participation are roughly consistent with the symmetric MSNE, individual subjects change their participation decision much less frequently. The histogram of the number of individual switches, depicted in the lower left panel of Fig. 2 also provides compelling evidence that many subjects are quite reluctant to change. Instead of playing according to the symmetric MNSE, subjects seem to condition their decision on past payoffs. The last row of Table 1 shows that about 73% of individual switches were preceded directly by a negative payoff signal (i.e., start (stop) to participate when that gave a higher (lower) payoff in the previous period). A last observation worth noting is the evidence for a 'participation premium' in the lower right panel of Fig 2. Subjects that participated more often did, on average, earn higher payoffs. We will return to this issue in Section 6.

3 A behavioral model

The experiment discussed above suggests, together with the earlier experimental evidence, that our negative feedback participating game is inherently unstable, with persistent fluctuations in the participation rate. Moreover, individual subjects do not seem to randomize their decisions and these fluctuations therefore cannot be easily attributed to mixed strategy Nash equilibria. Instead, subjects seem to base their decisions on deterministic behavioral rules. The existing behavioral models of negative feedback participation games are complex computational models, and some of them (such as reinforcement learning) do assume that agents randomize. In this section we introduce a behavioral model which is analytically tractable and consistent with the experimental findings.

3.1 Behavioral rules

Consider an infinite population of agents, which is randomly matched in groups of N players in every period to play the participation game introduced in the previous section. Each of the players is programmed to play the game according to one of K different behavioral rules. Each rule prescribes, conditional on the aggregate information, whether the agent using that rule should participate or not. Let x_t be the population-wide participation rate, i.e. the fraction of players in the entire population that participates in period t. A behavioral rule has the following form

$$p_{k,t} = f_k\left(\mathcal{I}_{t-1}\right), \tag{5}$$

where the information set is given by past participation rates:

$$\mathcal{I}_{t-1} = \left\{x_{t-1}, x_{t-2}, \ldots, x_1, x_0\right\}.$$

and where $p_{k,t} \in \{0, 1\}$.[4] Examples of simple rules are:

$$p_{1,t} = 1,\ p_{2,t} = 0 \text{ and } p_{3,t} = \begin{cases} 1 & \text{if} \quad x_{t-1} < b \\ 0 & \text{if} \quad x_{t-1} \geq b \end{cases}.$$

Rule 1 prescribes to participate always, whereas rule 2 specifies to never participate. According to rule 3 the player should participate if and only if the participation rate in the previous period turned out to be lower than the threshold value b. In fact,

[4]Notice that it would be straightforward to extend the setup to 'randomizing' decision rules, by allowing $p_{k,t}$ to be a number between 0 and 1, which then specifies the probability with which the player participates.

rule 3 is a special case of a *best-response rule*. Generally, a best-response rule can be written as

$$p_{k,t} = BR\left(g_k\left(x_{t-1}, x_t, \ldots, x_0\right)\right), \tag{6}$$

where $g_k(\cdot)$ is a predictor for the time t participation rate and BR corresponds to the best-responses to that prediction: $BR(g_k) = 1$ if and only if $g_k < b$, otherwise $BR(g_k) = 0$.

3.2 Evolutionary competition

A behavioral rule that performs relatively well in period t is adopted by a larger fraction of the population of players in period $t+1$. Let $x_{k,t}$ denote the fraction of the population using rule k in period t. The vector $\mathbf{x}_t \in \triangle^{K-1} = \left\{\mathbf{x}_t \in I\!R_+^K : \sum_{k=1}^K x_{kt} = 1\right\}$ then gives the distribution of agents over the rules. Notice that at time t aggregate participation x_t is completely characterized by $\mathbf{x}_t$ and $\mathbf{p}_t = (p_{1,t}, \ldots, p_{K,t})$. In fact, we have $x_t = \mathbf{x}_t \cdot \mathbf{p}_t = \sum_{k=1}^K x_{k,t} p_{kt}$.

The distribution $\mathbf{x}_t$ and the behavioral rules $\mathbf{p}_t$ induce a probability distribution over payoffs $(\alpha - \gamma, \alpha, \alpha + \beta - \gamma)$. Because we assume there are infinitely many players, the realized payoff $\pi_{k,t}$ of the players using rule k is equal to its ex-ante expected payoff. The payoff for rule k is given by $1 - p_k$ times α plus p_k times the payoff from participating, which is $\pi_p = \alpha + \beta p(x; N_c, N) - \gamma$, where $p(x; N_c, N)$ is again (see (3)) the probability that the number of other agents participating is less than N_c, given that the participation rate is x. We therefore get[5]

$$\pi_k(x; N_c, N) = \alpha + \left(\beta p\left(x; N_c, N\right) - \gamma\right) p_k. \tag{7}$$

The evolution of the distribution of rules, characterised by the vector $\mathbf{x}_t$, is determined by the payoffs $\pi_{k,t}$ generated by these rules. Given $\pi_t = (\pi_{1,t}, \pi_{2,t}, \ldots, \pi_{k,t})$ evolutionary competition between the rules is specified by

$$\mathbf{x}_{t+1} = \mathbf{H}\left(\mathbf{x}_t, \pi_t\right) \tag{8}$$

where $\mathbf{H} : \triangle^{K-1} \times [\alpha - \gamma, \alpha + \beta - \gamma]^K \rightarrow \triangle^{K-1}$ is assumed to be a continuous and differentiable function with $\frac{\partial H_k}{\partial \pi_k} > 0$ and $\frac{\partial H_j}{\partial \pi_k} < 0$ for $j \neq k$. It is straightforward to extend this evolutionary model by including lagged values $\mathbf{x}_{t-\tau}$ and $\pi_{t-\tau}$, for $\tau \geq 1$. An equilibrium of the evolutionary process (8) is a vector $(\mathbf{x}^*, \mathbf{p}^*) = (x_1^*, \ldots, x_K^*, p_1^*, \ldots, p_K^*)$, such that $\mathbf{x}^* = \mathbf{H}(\mathbf{x}^*, \pi^*)$. The equilibrium participation rate is then given by $x^* = \mathbf{x}^* \cdot \mathbf{p}^*$. In this case $p_k^* = f_k(x^*, \ldots, x^*)$ and equilibrium profits are given by $\pi_k^* = \pi_k(\mathbf{x}^*, \mathbf{p}^*; N_c, N)$.

There exist different specifications for (8) in the literature. In this paper we consider the replicator dynamics (see e.g. Taylor and Jonker, 1978, and Weibull, 1995), which is probably the most widely used updating mechanism in evolutionary economic dynamics and which takes the following format

$$x_{k,t+1} = \frac{x_{kt}\pi_{kt}}{\sum_j x_{jt}\pi_{jt}}. \tag{9}$$

[5]Note that since there is a one-to-one correspondence between x and π_p, the information set $\mathcal{I}_t$ is equivalent with $\{\pi_{p,t}, \pi_{p,t-1}, \ldots, \pi_{p,0}\}$.

Such an updating mechanism can be interpreted as a biological reproduction model, where each period the number of agents using rule k grows proportionally to the performance of that rule, as measured by its payoff π_k. Nevertheless (9) also arises in imitation processes in large populations of interacting agents (see e.g. Chapter 4 of Weibull, 1995).

Using (7), (9), $x_t = \sum_j x_{j,t} p_{j,t}$ and $\overline{\pi}_t = \sum_j x_{j,t} \pi_{j,t}$ we can write $\Delta x_{k,t+1} = x_{k,t+1} - x_{k,t}$ as

$$\Delta x_{k,t+1} = \frac{x_{k,t} \left(\pi_{k,t} - \overline{\pi}_t \right)}{\overline{\pi}_t} = \frac{x_{k,t} \left(\beta p \left(x_t; N_c, N \right) - \gamma \right) \left(p_{k,t} - x_t \right)}{\alpha + \left(\beta p \left(x_t; N_c, N \right) - \gamma \right) x_t}. \tag{10}$$

Steady states of the replicator dynamics (9) correspond to zeros of (10) for all k. Notice that there are K trivial steady states with $x_k^* = 1$ for $k = k^*$ and $x_k^* = 0$ for $k \neq k^*$, that is, where only one rule is used in the population. Such a steady state typically does not correspond to a Nash-equilibrium of the one-shot participation game. All other steady states require that $\sum_j x_{j,t}^* \pi_{j,t}^* = \pi_{k,t}^*$ for all k with $x_k^* > 0$. One type of these steady states is characterised by x^* such that $\beta p(x^*; N_c, N) = \gamma$. We will denote those steady states, which correspond to the symmetric MSNE, *generic*. There might also be *non-generic* steady states, where $p_k(x^*, \ldots, x^*) = p^*$ for all k with $x_k^* > 0$, such that $x^* = p^*$. These non-generic steady states only arise in the special case where the rules all intersect at a point x^* when evaluated at that point x^*. We have now proven the following result.

Proposition 2 *The system given by (5), (7) and (9) has K fixed points for which the whole population of players uses the same rule. Other fixed points satisfy the property that $\pi_k^* = \pi_{k'}^*$ for all k and k' such that $x_k^*, x_{k'}^* > 0$. In the generic case $x^* = s^*$, where s^* is the symmetric MSNE.*

In the next section we investigate under which conditions the replicator dynamics converges to the symmetric mixed strategy Nash equilibrium, and whether the replicator dynamics can describe the experimental and computational results discussed before.

4 Persistent participation fluctuations

Consider the model of evolutionary competition with the following two rules

$$p_{1,t} = 1 \text{ for all } t \tag{11}$$

$$p_{0,t} = 0 \text{ for all } t. \tag{12}$$

The "optimistic" rule 1 specifies to always participate and the "pessimistic" rule 0 prescribes to never participate. The experiment discussed in Section 2 suggests that many subjects use one of these rules for a substantial number of periods. Rules (11) and (12) can also be understood as best-reply rules (6), with (11) the best reply to an optimistic predictor (always predicting below b), and (12) the best-reply rule to a pessimistic predictor.

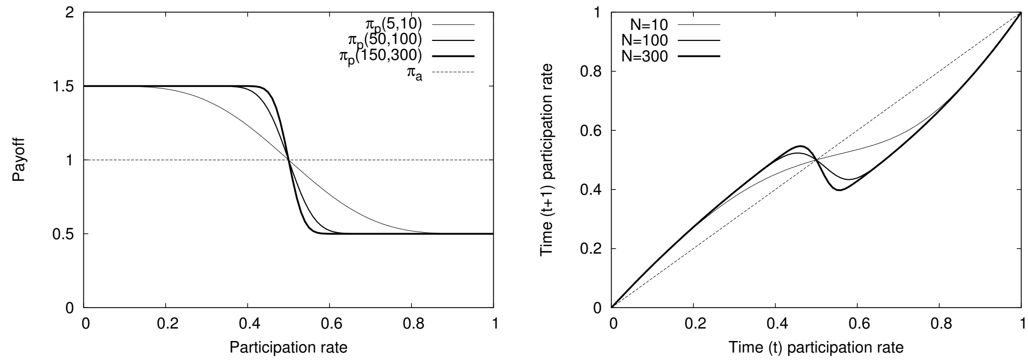

Figure 3: **Left panel**: Payoffs of the pessimistic rule (12) (horizontal line) and optimistic rule (11) for different values of N. **Right panel**: Graph of (14) for three different values of N. In both panels the other parameters are $\alpha = 1$, $\gamma = \frac{1}{2}$, $\beta = 2\gamma$ and $b = \frac{1}{2}$.

Using (9) the fraction of optimists evolves as

$$x_{1,t+1} = \frac{x_{1,t}\pi_{1,t}}{x_{1,t}\pi_{1,t} + x_{0,t}\pi_{0,t}}, \tag{13}$$

where $\pi_{0,t} = \alpha$ and $\pi_{1,t} = \alpha + \beta p\left(x_t; bN, N\right) - \gamma$. The left panel of Fig. 3 shows examples of the payoffs of the two rules as a function of x, for $b = \frac{1}{2}$ and different values of N. As N becomes large, the optimistic payoff $\pi_{1,t}$ converges to the step payoff function given in (1). On the other hand when N is small the expected payoff function becomes less steep.

Notice that since $x_t = x_{1,t} = 1 - x_{0,t}$, (13) can be written as

$$x_{t+1} = f\left(x_t; b, N\right) = \frac{x_t(\alpha + p(x_t; bN, N)\beta - \gamma)}{x_t(p(x_t; bN, N)\beta - \gamma) + \alpha}. \tag{14}$$

This first order nonlinear difference equation is parametrized by N, $b = \frac{N_c}{N}$ and the payoff parameters α, β and γ. The right panel of Fig. 3 shows (14) for different values of N given fixed values of the other parameters.

First notice that, since $p_1 \neq p_0$ there are no non-generic steady states. By Proposition 2 and inspection of the right panel of Fig. 3 we find that the steady states are given by $x = 0$, $x = 1$ and the symmetric MSNE $x = s^*$. To characterize local stability of these steady states it is convenient to define

$$\delta\left(x; b, N\right) = \frac{\partial p\left(x; bN, N\right)}{\partial x}.$$

Obviously $\delta\left(x; b, N\right) < 0$ since an increase in the fraction of agents participating always decreases the probability that less than bN of them indeed participate.

The following proposition characterises the stability properties of the steady states.

Proposition 3 *The dynamics of the participation rate given by (14) has three steady states:* 0*,* s^* *and* 1*. The steady states* 0 *and* 1 *are locally unstable. The interior steady state* s^* *is locally stable when* $\psi \equiv s^*\left(1 - s^*\right)\frac{\delta^*\beta}{\alpha} > -2$*, where* $\delta^* \equiv \delta\left(s^*; b, N\right)$*.*

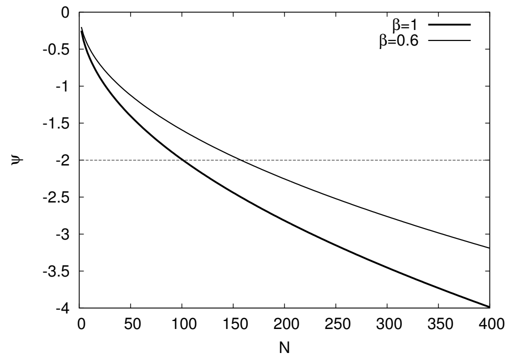

Figure 4: **Stability condition**. ψ as a function of N for $\alpha = 1$, $\gamma = \frac{1}{2}\beta$, $b = \frac{1}{2}$ and two values of β: $\beta = 0.6$ (upper curve) and $\beta = 1$ (lower curve).

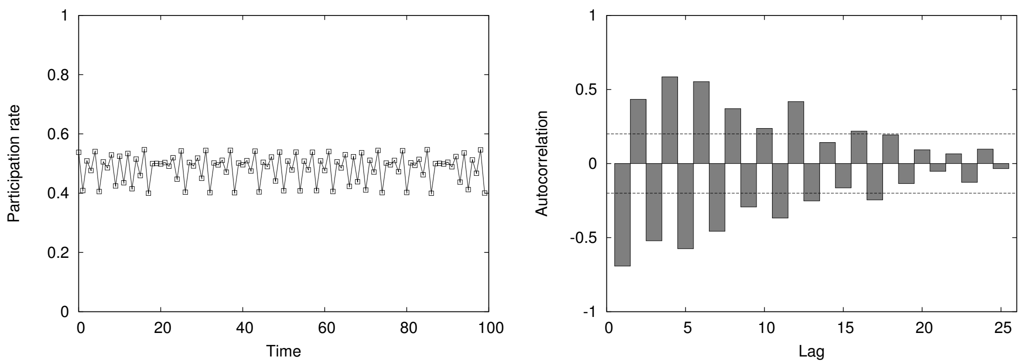

Figure 5: **Left panel**: Time series of the participation rate. **Right panel**: Autocorrelation diagram. Parameters values are $N = 300$, $N_c = 150$, $\alpha = 1$, $\gamma = \frac{1}{2}$ and $\beta = 2\gamma$.

The stability condition in Proposition 3 depends, through δ^* and s^*, implicitly on b and N. We are particularly interested in characterising this stability of s^* as a function of N. Fig. 3 already suggests that s^* becomes unstable for N large enough. The following proposition corroborates that.

Proposition 4 *For any given value of α, $\gamma < \beta$, and b the dynamics of the participation rate is locally unstable around s^* in the limit $N \to \infty$. Moreover, if $\beta = 2\gamma$ and $b = \frac{1}{2}$ there exists a unique M such that equation (14) is locally stable around s^* if and only if $N < M$.*

Fig. 4 shows that ψ is decreasing in N and crosses -2 when N is larger than some treshold M. The intuition behind Proposition 4 is that as N increases the average population payoff of the optimistic rule gets closer to the step payoff of the underlying one shot game (see left panel of Fig. 3). As a result, for any value of the payoff parameters, $\alpha, \beta > \gamma$, as N increases (14) becomes steeper at the steady state s^* and the system looses stability. This dependence of the dynamics upon N

is due to the assumption of random matching. In fact, for $N = 2$ (with $N_c = 1$) the expected payoff function is linear in x, since $p(x; 1, 2) = 1 - x$, which is the probability of meeting a player using the pessimistic rule. However, the probability of having less than half of the players participating when each player participates (or, is an optimist) with probability larger than $\frac{1}{2}$ becomes small as N becomes large. That is, as N increases (and for given b) the function $p(x; N_c, N)$ will look more and more like a step function. We can therefore also interpret the parameter N as a measure of the shape and steepness of the payoff function at the steady state. In that case a low value of N would present a payoff function which decreases slowly as the number of participating players increases, similar to the the linear one used in the early market entry experiments (see, for example, Sundali *et al.*, 1995). A high value of N, on the other hand, would represent an expected payoff function close to the step function used in the *El Farol* bar game, with payoffs at the symmetric MSNE dropping rapidly as an extra player participates.

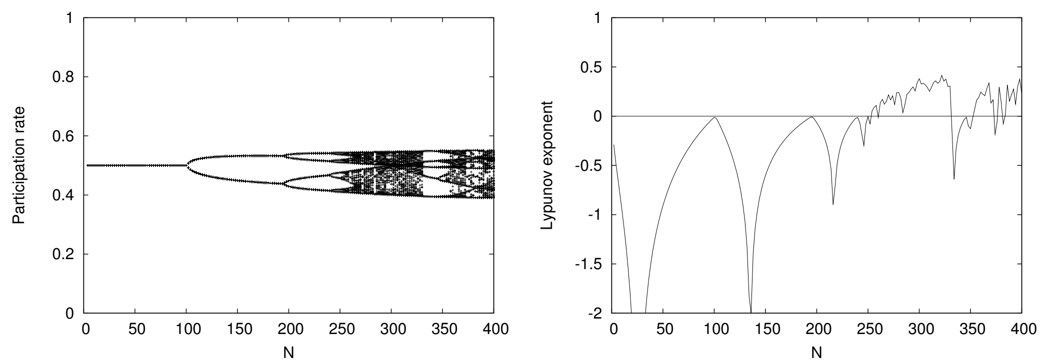

Figure 6: **Left panel**: Bifurcation diagram with respect to N. **Right panel**: Lyapunov exponents for different values of N. Parameters values are $b = \frac{N_c}{N} = \frac{1}{2}$, $\alpha = 1$, $\gamma = \frac{1}{2}$ and $\beta = 2\gamma$. For every value of N, 100 iterations are shown after an initialization period of 100.

Now we consider the global dynamics of our evolutionary model when the steady state s^* is locally unstable. For the benchmark case with $b = 1/2$ and $\beta = 2\gamma$ the critical value of M from Proposition 4 is given exactly by $\widetilde{M} = 100$. Fig. 5 shows the participation rate generated by (14) when $N = 300$ (and consequently $N_c = bN = 150$). The time series (left panel) looks aperiodic. In fact, for certain parameter regions the dynamics is chaotic.[6] Even such a simple one-dimensional system is therefore able to produce complicated times series similar to those obtained by the computationally intensive simulation models of Arthur (1994) and others. The left panel of Fig. 6 shows, for every even value of N between 2 and 400, the effect on the dynamics of x_t (after a suitable initialization period). Clearly, as N increases the dynamics of the participation rate x becomes unstable (at $N = \widetilde{M} = 100$),

[6]The left panel of Fig. 7 shows that for some parameter values (for example, for $\beta - \gamma \approx 0.6$) this one-dimensional system has a 3-cycle which implies, by the Li-Yorke theorem (Li and Yorke, 1975) that the dynamics for those parameter values is chaotic.

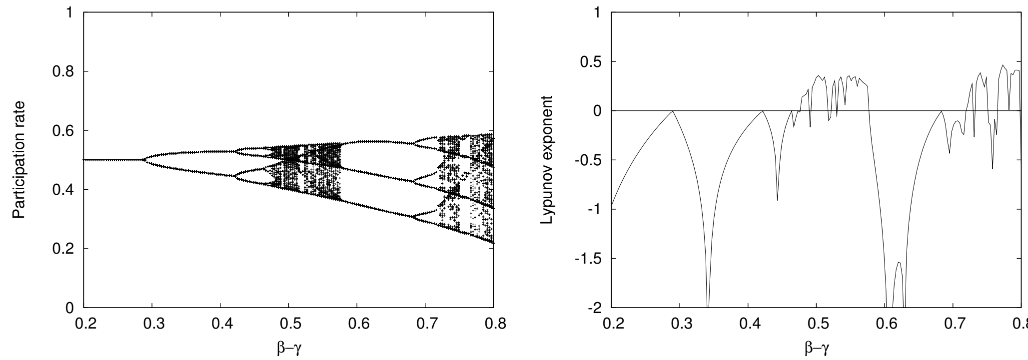

Figure 7: **Left panel**: Bifurcation diagram with respect to N. **Right panel**: Lyapunov exponents for different values of N. Parameters values are $b = \frac{N_c}{N} = \frac{1}{2}$, $\alpha = 1$, $\gamma = \frac{1}{2}$ and $\beta = 2\gamma$. For every value of N, 100 iterations are shown after an initialization period of 100.

after which a period-doubling route to chaos sets in. The right panel of Fig. 6 shows the Lyapunov exponents of the corresponding participation rate for each N. A positive Lyapunov exponent (e.g. for $N = 300$) characterizes sensitive dependence on initial conditions and implies that the system is chaotic for that parameter setting. Fig. 7 shows the dynamics and Lyapunov exponents for different values of $\beta - \gamma$. The right panel shows that there are many values of $\beta - \gamma$ for which the dynamics is chaotic. The left panel suggests that fluctuations increase, and thus allocative efficiency decreases, as $\beta - \gamma$ increases. Nevertheless, for given $\beta - \gamma$ the size of the fluctuations of the participation rate seems to be constant in N for N larger than, say, 250 players (see left panel of Fig. 6). Note that at the symmetric MSNE the fluctuations in the participation rate go to zero as the number of players increases, since each of these players is randomizing independently. Experiments with different values of N, but constant b, may shed light on the relationship between N and the size of the fluctuations. As a final observation note that if the dynamics are unstable the average participation rate is below s^*. This 'underparticipation' is most apparent in the left panel of Fig. 7, but can also be found in the other numerical results.

5 The scope for arbitrage

In the previous section we established that for large N the evolutionary competition between an optimistic and a pessimistic rule induces fluctuations of the aggregate participation rate x around the steady state s^*. The right panel in Fig. 5 shows that there is a significant correlation between the participation rates and its lagged values. The participation rate series is therefore not informationally efficient and one might argue that the competition between the optimistic and pessimistic rule is not evolutionary stable against other rules. The aim of this section is to investigate whether new rules may succeed in exploiting these regularities and how the inclusion of such new rules affects the participation rate dynamics. Similar analyses have been

performed in Hommes (1998) and Brock *et al.* (2006) for the cobweb model.
Since there is a strong positive autocorrelation at the second lag, indicating 'up-and-down' behavior, a best response to predicting x_{t-2} for period t seems to be a sensible strategy. This so-called "two lags best reply" rule reads[7]

$$p_{2,t} = BR(x_{t-2}) = \begin{cases} 1 & \text{if} \quad x_{t-2} < b \\ 0 & \text{if} \quad x_{t-2} \geq b \end{cases}, \tag{15}$$

Letting $x_{2,t}$ be the fraction of two lags best responders, the participation rate at time t is given as:

$$x_t = x_{1,t} + x_{2,t} BR(x_{t-2}),$$

where $x_{1,t}$ and $x_{0,t} = 1 - x_{1,t} - x_{2,t}$ are the fractions of optimists and pessimists, respectively. As before, the latter have payoffs α and expected payoff for optimists is given by $\pi_p(x_t) = \alpha + \beta p(x_t; N_c, N) - \gamma$. Expected payoffs for two lags best responders are:

$$\pi_{2,t} = (1 - BR(x_{t-2}))\alpha + BR(x_{t-2})\pi_p(x_t).$$

Fractions $x_{0,t+1}$, $x_{1,t+1}$ and $x_{2,t+1}$ develop according to the replicator dynamics (9), which is based upon payoffs generated by the different rules, which, in turn, depend upon x_t and x_{t-2}. This leads to a four-dimensional dynamical system. The interior steady state of this system is characterised by $x = s^*$. At $x = s^*$ every rule generates the same expected payoff. If $s^* \geq b$ (as in our benchmark specification with $s^* = b = \frac{1}{2}$) the two lags best responders are not participating at the steady state. Therefore, there is a continuum of steady state fractions with $x_1^* = s^*$ and $x_0^* + x_2^* = 1 - s^*$. Given that the rule $p_{2,t}$ is discontinuous we rely on numerical simulations to determine local stability of these steady states. The evolutionary model with these rules is unstable for the same values of N as before (that is, when $N > \widetilde{M} = 100$) and leads to persistent fluctuations in participations rates for N large enough. The left panel of Fig. 8 shows the time average of the fraction of two lagged best responders along 100 iterations. This average fraction approaches zero when the steady state s^* is locally stable and is strictly positive otherwise. The reason for this is that for $N < \widetilde{M}$ the dynamics converges to the steady state $x = s^*$ along a "shrinking" 3-cycle. Along such a cycles a two lagged best responder is more often wrong in its prediction than right and slowly disappears. Notice that, even if x_2 approaches zero, it could happen that the participation rate x settles at s^* before x_2 equals zero. The simulations show that this typically does not happen. A different participation rate dynamics occurs when $N > \widetilde{M}$. In this case the dynamics is typically non-periodic and two lags best responders survive. The upper left panel of Fig. 9 shows the resulting participation rate for $N = 300$. Even though the time series is not periodic, the autocorrelation diagram (upper right panel of Fig. 9) shows that the autocorrelation at the second lag has indeed decreased substantially (and in fact is not significantly different from zero). However, the autocorrelation at the third lag is strongly positive now, suggesting that the time series of participation rates has elements of a noisy 3-cycle.

[7] The reader may wonder why we do not investigate a "one lag best reply" rule. Since the interaction between the optimistic and the pessimistic rule generates a strong first order negative autocorrelation in participation rates, one lag best responders are very often wrong in their prediction. Simulations confirm that these one lag best responders are quickly driven out.

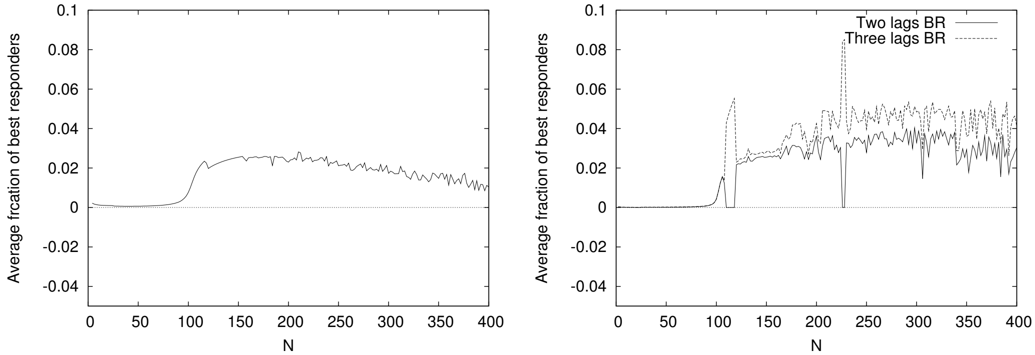

Figure 8: **Left panel**: Fraction of two lags best responders in an competition between optimists, pessimists and two lags best responders, for different different N and averaged over 100 periods. **Right panel**: Fraction of two and three lags best responders in competition between optimists, pessimists, two and three lags best responders, for different values of N and averaged over 100 periods. Other parameter values: $b = \frac{1}{2}$, $\alpha = 1$, $\beta = 2\gamma$, $\gamma = \frac{1}{2}$.

This begs the question as to what would happen if a new rule that tries to exploit this feature is introduced. To that end, we introduce the "*three* lags best reply" rule $p_{3,t} = BR(x_{t-3})$. The evolutionary competition of the four rules gives a six-dimensional dynamical system. Given that $s^* \geq b$ there is a continuum of steady states of the type $x_1 = s^*$ and $x_0 + x_2 + x_3 = 1 - s^*$. The steady state s^* is unstable again for $N > \widetilde{M} = 100$ and leads to erratic participation rates for high values of N, as can be seen in the middle left panel of Fig. 9. The right panel of Fig. 8 shows the average fraction (over 100 iterations) of two and three lags best responders. When $N < \widetilde{M}$ the participation rate converges to s^* and the fractions of both types of best responders go to zero. This is because the participation rate converges to $x = s^*$ along 4-cycles of decreasing amplitude, along which both types of best responders are more often wrong than right in their predictions. Instead, when $N > \widetilde{M}$, the dynamics is unstable and both types of best responders may survive as can be seen. It is interesting to see that typically these best response rule are driven out when the dynamics is stable so that only the optimistic and pessimistic rule survive. The autocorrelation diagram depicted in the middle right panel of Figure 9 suggests that x_{t-4} would be a good predictor of x_t. Again therefore, the evolutionary competition leads to a regularity in participation rates that cannot be exploited by the rules that are present in the population. Adding new rules does not stabilize the dynamics, it merely drives out one regularity at the expense of introducing higher-order regularities. This is illustrated by the lower left panel of Fig. 9 where the results are shown of an evolutionary model with the optimistic rule, the pessimistic rule, and two, three, four, five and six lags best responders, for $N = 300$. The participation rate still exhibits perpetual fluctuations, in this case with positive autocorrelation at the 7th lag. Autocorrelation at lower lags is exploited efficiently by the existing behavioral rules and is not significantly different from zero.

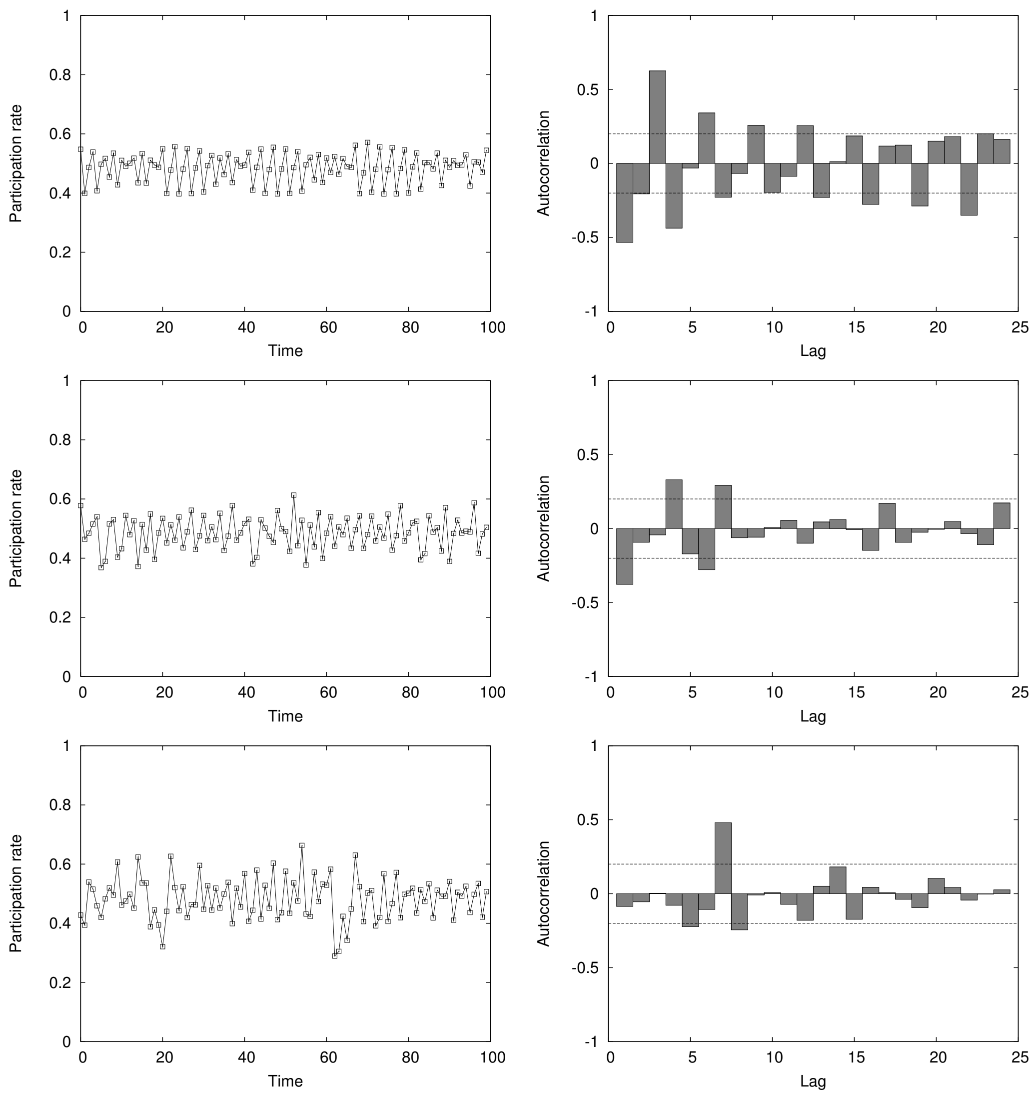

Figure 9: **Upper panels**: Participation rate and autocorrelation diagram for competition between optimists, pessimists and two lag best responders. **Middle panels**: Participation rate and autocorrelation diagram for competition between optimists, pessimists two and three lag best responders. **Lower panels**: Participation rate and autocorrelation diagram for competition between optimists, pessimists and two to six lag best responders. The dotted lines show the significance level of 100 observations. Parameters values $N = 300$, $b = \frac{1}{2}$, $\alpha = 1$, $\beta = 2\gamma$ and $\gamma = \frac{1}{2}$.

This numerical analysis shows that introducing rules that try to exploit a particular time series structure indeed makes sure that cycles are arbitraged away but they do not necessarily stabilize the dynamics. In fact, adding rules typically complicates the dynamics and makes it more unpredictable.

6 The participation premium

Both the market entry experiment discussed in Section 2 and the behavioral model from Sections 4 and 5 exhibit underparticipation, that is, the average participation rates are below the steady state value s^*, when this steady state is unstable. This underparticipation may result in a 'participation premium': the lower right panel of Fig. 2, for example, shows that subjects in the experiment that participate more often typically earn higher average payoffs. We conjecture that this participation premium results from the payoff-asymmetry of the two alternatives in the market entry game. That is, players choose between a strategically uncertain alternative (participating), for which the payoff depends upon the actions of the other players and a sure alternative (not participating), where the payoff is independent of other player's actions. In this section we consider route choice games to investigate this conjecture. For the route choice game both alternatives are subject to strategic uncertainty and payoffs always depend upon the choices of the other players.
In the route choice game the payoff of player i for participating, $\pi_i\left(1, \mathbf{a}_{-i}; N_c, N\right)$, is still given by (1), but the payoff for not participating changes into

$$\pi_i\left(0, \mathbf{a}_{-i}; N_c, N\right) = \begin{cases} \alpha + \beta - \gamma & \sum_{j \neq i, j=1}^{N} a_j \geq N_c \\ \alpha - \gamma & \sum_{j \neq i, j=1}^{N} a_j < N_c \end{cases} .$$

There are $\binom{N}{N_c}$ pure strategy Nash equilibria (PSNE) where exactly N_c players participate. These Nash equilibria are not strict: a player not participating is indifferent between $a = 0$ and $a = 1$. In fact, there are $\binom{N}{N_c+1}$ other PSNE where exactly $N_c + 1$ players participate. These equilibria are also not strict since a participating player is now indifferent between participating and not. Consequently there is an infinite number of asymmetric MSNE. There is only one symmetric MSNE, which is given by s^* such that $p\left(s^*; N_c, N\right) = \frac{1}{2}$. In contrast to the market entry game (see condition (4)) the symmetric MSNE s^* is independent of the ratio γ/β. Note that here, for any value of α and for any $\gamma < \beta$, the probability s^* has to be such that the distribution of the participation rate of $N-1$ players has $N_c - 1$ as its median.
In Fig. 10 we present the dynamics of the participation rate resulting from the interaction of optimists (always choosing alternative 1), pessimists (always choosing alternative 0) and 'two lag best responders'. As before, two lag best responders have an effect on the dynamics only when the steady state is locally unstable. Furthermore, fluctuations seem to be symmetrically distributed around s^*, and underparticipation indeed disappears. Also notice that the condition for the local stability is more stringent here than for the market entry game, since the system loses stability at a much lower value of N.
The following result may help us in understanding the relationship between underparticipation and the asymmetric payoff structure of the market entry game.

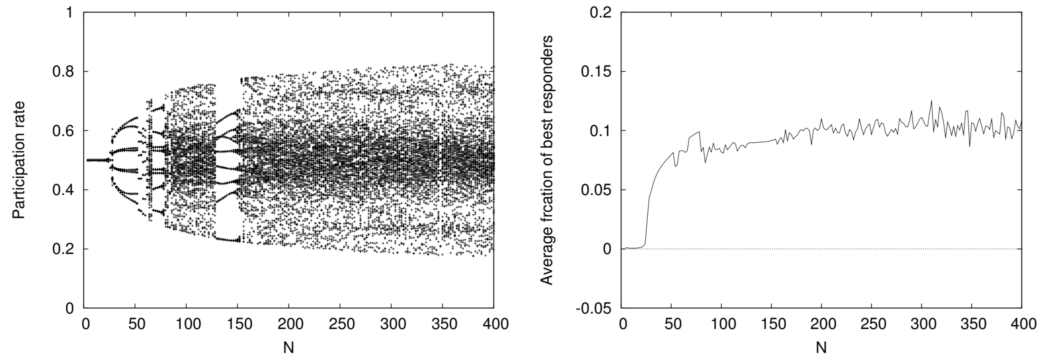

Figure 10: Evolutionary competition of optimists, pessimists and two lags best responders in a route choice setting. **Left panel**: Bifurcation diagram showing the long run behavior of the participation rate for different values of N. **Right panel**: 100 periods time average of the fraction of two lags best responders for different values of N. The time average is computed along the iterations shown in the left panel. Parameters are $b = \frac{1}{2}$, $\alpha = 1$, $\beta = 2\gamma$, $\gamma = \frac{1}{2}$.

Proposition 5 *Consider an evolutionary competition between the optimistic and pessimistic rule and let $b = 1/2$ and $\beta = 2\gamma$. Denote by $z_t = x_t - s^*$ the deviations of the participation rate from the MSNE and define $\Delta z_{t+1} = z_{t+1} - z_t$. For the market entry game we can write $\Delta z_{t+1} = m(z_t)$, where $m(z) + m(-z) < 0$, for $z \notin \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$. For the route choice game we can write $\Delta z_{t+1} = r(z_t)$, where $-r(z) = r(-z)$.*

The left panel of Fig. 11 shows the functions $m(z)$ and $r(z)$ and illustrates that, outside the three steady states $z = -\frac{1}{2}$, $z = 0$ and $z = \frac{1}{2}$, we have $m(z) < r(z)$. This, together with Proposition 5 suggests that there is a tendency for z_t to be downward biased in the market entry game, since innovations in z are lower than in the route choice game (this is corroborated for example in the left panel of Fig. 7). The origin of the 'asymmetry' of $m(z)$ lies in the denominator of (10), the population average payoff $\overline{\pi}_t = \sum_i x_{i,t}\pi_{i,t}$. The right panel of Fig. 11 shows that average payoffs $\overline{\pi}$ are symmetric in x around $s^* = \frac{1}{2}$ for the route choice game, but not for the market entry game.

This asymmetry in $\overline{\pi}$ leads to underparticipation in the market entry game, and a participation premium for those agents using the optimistic rule, since the project is more often profitable. This is illustrated by the left panel of Fig. 12 which shows the difference between average payoff of optimists and pessimists for the market entry and the route choice game. For the route choice game average payoffs of optimists and pessimists are, due to symmetry, always the same. For the market entry game however, just as in the experiment discussed in Section 2, optimists do better on average, whenever the steady state s^* is unstable.

One might expect that this payoff difference in the market entry game disappears when some "memory" is introduced. In fact when memory plays a role more agents should imitate the action of the optimists, which are performing better, and thus eliminate underparticipation. Consider the evolutionary competition of optimists

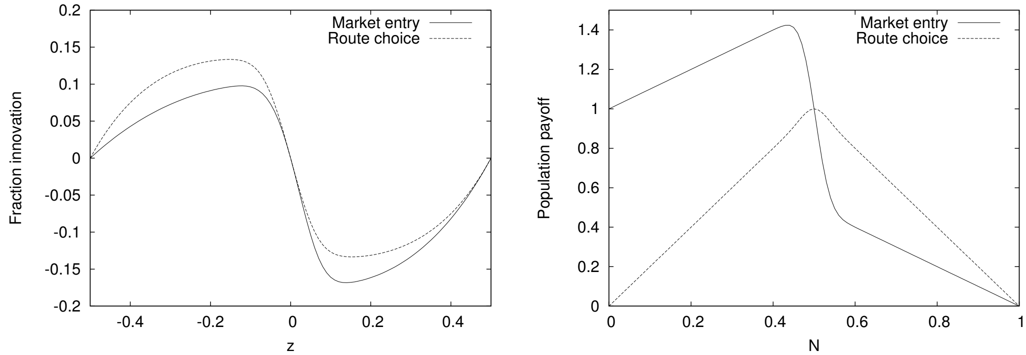

Figure 11: Comparison of the evolutionary competition of optimists and pessimists for market entry games and route choice games. **Left panel**: Innovation function for the market entry game and for the route choice game compared. **Right panel**: Population average payoffs, $\bar{\pi}$, in the two cases. Parameters are $b = \frac{1}{2}$, $\alpha = 1$, $\beta = 2\gamma$ and $\gamma = \frac{1}{2}$.

versus pessimists where evolution is governed by a fitness measure that is a weighted average of past payoffs, that is, $F_t = \mu F_{t-1} + (1-\mu)\,\pi_{1,t-1}$. The resulting dynamical system is two dimensional, with variables x_t and F_t, and given by:

$$\begin{aligned} x_{t+1} &= f_1(x_t, F_t) = \frac{x_t F_t}{x_t F_t + (1-x_t)\,\alpha} \\ F_{t+1} &= f_2(x_t, F_t) = \mu F_t + (1-\mu)\left(\alpha - \gamma + \beta p\left(f_1(x_t, F_t); N_c, N\right)\right) \end{aligned} \tag{16}$$

The following proposition characterizes its behavior.

Proposition 6 *The dynamics of the participation rate and of the optimists' payoff given by the system (16) has three steady states:* $(0, \alpha - \gamma + \beta)$, (s^*, α) *and* $(1, \alpha - \gamma)$. *The steady states* $(0, \alpha - \gamma + \beta)$ *and* $(1, \alpha - \gamma)$ *are locally unstable. The interior steady state* (s^*, α) *is locally stable when* $\psi \equiv s^*(1-s^*)\frac{\delta^* \beta}{\alpha} > -2\frac{1+\mu}{1-\mu}$.

The effect of memory is to stabilize the dynamics. For $\mu = 0$ we retrieve the stability condition from Proposition 3, but as μ increases the critical value of ψ increases. Nevertheless, when the dynamics is unstable the same difference in time averages payoffs as before emerges. The right panel of Fig. 12 shows that when the steady state s^* is unstable over time optimists outperform pessimists. Therefore, the participation premium persists.

Summarizing, the asymmetric payoff-structure of the market entry game is indeed responsible for underparticipation and the participation premium. Note that if we assume agents are driven by payoff differences instead of absolute payoffs, the market entry game transfers naturally into a route choice problem. To see this, consider the payoff function (1) and define the payoff difference by $\phi_i(a_i, \mathbf{a}_{-i}; N_c, N) = \pi_i(a_i, \mathbf{a}_{-i}; N_c, N) - \pi_i(1-a_i, \mathbf{a}_{-i}; N_c, N)$. This payoff difference is equal to $\beta - \gamma$ or $-\gamma$ for participating players, and equal to $\gamma - \beta$ or γ for players that do not participate, depending on the number of participating players. Clearly, $\phi_i(1, \mathbf{a}_{-i}; N_c, N) +$

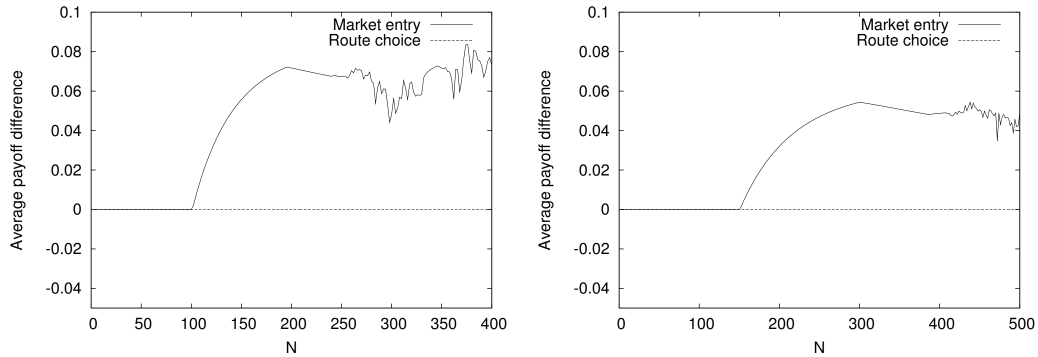

Figure 12: Optimists - pessimists time average payoff differences for market entry games and route choice games. **Left panel**: Time average payoff difference without memory. **Right panel**: Time average payoff difference with memory. Parameters are $b = \frac{1}{2}$, $\alpha = 1$, $\beta = 2\gamma$, $\gamma = \frac{1}{2}$ and $\mu = \frac{1}{10}$.

$\phi_i\left(0, \mathbf{a}_{-i}; N_c, N\right) = 0$ and choosing not to participate is subject to strategic uncertainty about payoff differences. Therefore, if players care about payoff differences between the alternatives participation rate dynamics are less stable and underparticipation is alleviated.

7 Conclusions

Many (economic) decision problems can be characterised as negative feedback participation games and understanding human behavior in these participation games is important. The experiment from Section 2 shows that, although a reasonable description at an aggregate level, the symmetric mixed strategy Nash equilibrium does not explain individual behavior very well. In particular, rather than randomizing their decisions, subjects typically use deterministic rules, possibly conditional on past outcomes. Moreover, the participation rate is inherently unstable, in all experimental groups. A series of contributions from computational economics, starting with the famous *El Farol* bar problem from Arthur (1994), also shows that complicated dynamics arise naturally in negative feedback participation games. Other complex and computationally intensive models assume that agents are randomizing their participation decision. In general, these models are difficult to study analytically and results from thise literature are typically based upon numerical simulations.

In this paper we introduce an alternative type of behavioral model that is able to explain the experimental and computational results, but still is sufficiently simple to be analyzed theoretically. We consider an evolutionary competition between different deterministic behavioral rules, where players switch between these rules on the basis of past performance. For the simplest possible case, where the only available two rules are those that specify to always participate, or to never participate, respectively, the participation rate dynamics evolves according to a nonlinear one-dimensional difference equation. This difference equation can be studied analytically, and local stability of the symmetric mixed strategy Nash equilibrium turns out to depend upon the number of players. For a large number of players this simple model exhibits perpetual fluctuations in the participation rate, similar to those found in the experiments and the, much more complex, computational models. A testable prediction of our model is that these fluctuations, in contrast to the mixed strategy Nash equilibrium, are not vanishing even when a very large group of players is involved.

The erratic time series of participation rates has two other features. First, the time series exhibits certain regularities. When rules that try to exploit this structure are introduced, this particular structure disappears, but fluctuations around the symmetric mixed strategy Nash equilibrium do not vanish. Instead, other (higher order) regularities are introduced. Again, adding more sophisticated behavioral rules drives out these regularities again, but does not stabilize the fluctuations, which therefore seem to be quite robust. Secondly, the time series exhibits underparticipation and a premium for participating. This is consistent with the experimental results. We establish that this is due to the asymmetry in the strategic uncertainty of the market entry game. This has interesting economic implications. In our future research we will try to use our behavioral model to explain certain economic or financial stylized facts, such as *excess volatility* and the so-called *equity premium puzzle* (Mehra and Prescott, 1985). Consider, for example, the decision to invest money in bonds, or in an index of stocks as an application of our model. The uncertainty of investing in the stock index is high and may depend on other agents choices, whereas investing in bonds is relatively safe. Our behavioral model predicts an excess return to investing in the stock index. This is consistent with the equity premium puzzle, which refers to empirical evidence that, after adjusting for risk, investing in stocks indeed is more profitable than investing in bonds.

Appendix: Proofs of the main results

Proof of Proposition 1 As argued in the text, the symmetric mixed strategy equilibrium s^* corresponds to the solution Eq. (4) from Section 2.The function $p(x; N_c, N)$ on the right-hand side of (4) is the cumulative distribution function (c.d.f.) of a binomial distribution, with $N-1$ degrees of freedom and probability of participating x, evaluated at $N_c - 1$. This implies that for every $N_c < N$, it holds true that $p(0; N_c, N) = 1$ and $p(1; N_c, N) = 0$. Furthermore $p(x; N_c, N)$ is continuous in x and $\frac{\partial p(x;N_c.N)}{\partial x} < 0$: if we increase the probability of participating the value of the c.d.f. at any fixed value between 0 and $N-1$ decreases. Consequently, since $\gamma < \beta$, $p(x; N_c, N) = \frac{\gamma}{\beta}$ has a unique solution s^* for any value of $N > 1$, any $N_c \in [0, N-1]$ and any $\gamma < \beta$. Furthermore, since equation (4) does not depend on α, neither does its solution s^*.

Typically $s^* \neq \frac{N_c}{N}$. We first show that when $N_c = \frac{1}{2}N$ and $\beta = 2\gamma$ then $s^* = \frac{N_c}{N} = \frac{1}{2}$.

Note that for all $N > 1$ we have

$$\begin{aligned}
1 &= \sum_{k=0}^{N-1} \binom{N-1}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{N-1-k} \\
&= \sum_{k=0}^{\frac{1}{2}N-1} \binom{N-1}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{N-1-k} + \sum_{k=\frac{1}{2}N}^{N-1} \binom{N-1}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{N-1-k} \\
&= p\left(\frac{1}{2}; \frac{1}{2}N, N\right) + \sum_{j=N-1-k=0}^{\frac{1}{2}N-1} \binom{N-1}{j} \left(\frac{1}{2}\right)^{N-1-j} \left(\frac{1}{2}\right)^j \\
&= p\left(\frac{1}{2}; \frac{1}{2}N, N\right) + p\left(\frac{1}{2}; \frac{1}{2}N, N\right) = 2p\left(\frac{1}{2}; \frac{1}{2}N, N\right).
\end{aligned}$$

which gives $p\left(\frac{1}{2}; \frac{1}{2}N, N\right) = \frac{1}{2} = \frac{\gamma}{\beta}$. Therefore $s^* = \frac{1}{2}$ is the unique MSNE for this specification.

In the final part of this proof we show that for general values of β, $\gamma < \beta$, N and $b = \frac{N_c}{N} \in (0,1)$ we have $s^* \to b$ as $N \to \infty$. Define the random variable $\mathbf{n} = \frac{\mathbf{N-1}}{N-1}$, where $\mathbf{N-1}$ is a random variable with Bernoulli distribution with probability s^* and $N-1$ degrees of freedom. Given the fact that s^* solves Eq. (4), the γ/β percentile of the distribution of $\mathbf{n}$ is given by $(bN-1)/(N-1) = b - (1-b)/(N-1)$. Also notice that the distribution of $\mathbf{n}$ has mean s^* and variance $s^*(1-s^*)/(N-1)$. Notice that when $N \to \infty$ the distribution of $\mathbf{n}$ is concentrated more and more around s^*. Assume there exists an $\varepsilon > 0$ such that when $N \to \infty$, $|b - s^*| \geq \varepsilon$. This implies that either $s^* > b$ or $s^* < b$. If $s^* > b$, then $\Pr(\mathbf{n} \leq b - (1-b)/(N-1)) \to 0$ when $N \to \infty$. This contradicts that s^* has been chosen such that the γ/β percentile of $\mathbf{n}$ is $b - (1-b)/(N-1)$. On the other hand, if $s^* < b$, then when $N \to \infty$, $\Pr(\mathbf{n} \leq b - (1-b)/(N-1)) \to 1$. This also contradicts that s^* is such that the γ/β percentile of $\mathbf{n}$ is given by $b - (1-b)/(N-1)$. We conclude that for every $\varepsilon > 0$, $|b - s^*| < \varepsilon$ as $N \to \infty$. □

Proof of Proposition 2 In text. □

To prove Propositions 3 and 4 the following result is useful.

Lemma 1 *Define,* $\delta^* = \delta(s^*; b, N,) = \left.\frac{dp}{dx}\right|_{x=s^*}$ *then, for a fixed value of* b*,* δ^* *is decreasing in* N *and* $\delta^* \to -\infty$ *as* $N \to \infty$.

Proof The cumulative distribution of a binomial distribution with parameters x and $N-1$ evaluated at $bN-1$ can be written in terms of the Beta-function, $B(x,y) = \frac{(x-1)!(y-1)!}{(x+y-1)!}$, as

$$p(x; bN, N) = 1 - \frac{\int_0^x t^{Nb-1}(1-t)^{N(1-b)-1}\,dt}{B(Nb, N(1-b))} \tag{17}$$

Equation (17) implies that we have

$$\delta\left(x;b,N\right)=\frac{\partial p\left(x;bN,N\right)}{\partial x}=-\frac{x^{Nb-1}\left(1-x\right)^{N(1-b)-1}}{B\left(Nb,N\left(1-b\right)\right)}$$

The function $\delta\left(x;b,N\right)$ has a unique maximum at $\widehat{x}_{b,N}=\frac{bN-1}{N-2}$. The associated minimum value of δ is given by get

$$\begin{aligned}\widehat{\delta}_{b,N}=\delta\left(\widehat{x}_{b,N};b,N\right) &= -\frac{(N-1)!}{(bN-1)!\left(N\left(1-b\right)-1\right)!}\left(\widehat{x}_{b,N}\right)^{bN-1}\left(1-\widehat{x}_{b,N}\right)^{N(1-b)-1} \\ &= -\frac{(N-1)\left(N-2\right)!}{(bN-1)!\left(N(1-b)-1\right)!}\left(\widehat{x}_{b,N}\right)^{bN-1}\left(1-\widehat{x}_{b,N}\right)^{N(1-b)-1}.\end{aligned}$$

Taking the logarithm of $-\widehat{\delta}_{b,N}$ and applying the Stirling approximation formula, $\log\left(n!\right)=n\log\left(n\right)-n+\xi(n)$ where $\xi(n)\rightarrow 0$ as $n\rightarrow\infty$, we getlog

$$\begin{aligned}\log\left(-\widehat{\delta}_{b,N}\right) &= \log\left(N-1\right)+\left(N-2\right)\log\left(N-2\right)-\left(N-2\right) \\ &\quad -\left(Nb-1\right)\log\left(\frac{Nb-1}{\widehat{x}_{b,N}}\right) \\ &\quad -\left(N\left(1-b\right)-1\right)\log\left(\frac{N\left(1-b\right)-1}{1-\widehat{x}_{b,N}}\right)+\left(N-2\right)+\xi\left(N\right) \\ &= \log\left(N-1\right)+\xi\left(N\right)\end{aligned}$$

which goes to ∞ as $N\rightarrow\infty$ with the same speed as $\log(N)$ and therefore $\widehat{\delta}_{b,N}\rightarrow-\infty$ as $N\rightarrow\infty$. For the special case $b=\frac{1}{2}$ we have $\widehat{x}_{b,N}=\frac{1}{2}$ for all values of N and, by Proposition 1, $s^*=\frac{1}{2}$ for all even values of N. This implies that for $b=\frac{1}{2}$ we have $\delta^*\rightarrow-\infty$ as $N\rightarrow\infty$. For the general case with $b\neq\frac{1}{2}$, both s^* and $\hat{x}_{b,N}$ converge to b as N goes to infinity. Moreover, $\delta\left(x;b,N\right)$ is conitnuous in x. Consequently it must be the case that also then $\delta^*\rightarrow-\infty$ as $N\rightarrow\infty$. □

Proof of Proposition 3 A simple computation shows that the equation $f\left(x;b,N\right)=x$, with $f\left(x;b,N\right)$ given by (14) has three steady states: $x=0$, $x=1$ and $x=s^*$, where s^* is the unique number solving $p\left(s^*;bN,N\right)=\gamma/\beta$. The derivative of (14) is given by

$$f'(x;b,N)=\frac{\alpha\left(\left(1-x\right)x\beta\delta+\left(\alpha+\beta p\left(x;bN,N\right)-\gamma\right)\right)}{\left(x_t\left(p\left(x_t;bN,N\right)\beta-\gamma\right)+\alpha\right)^2}\tag{18}$$

From (18) it follows immediately that $f'\left(1\right)=\frac{\alpha}{\alpha-\gamma}>1$ and $f'\left(0\right)=\frac{\alpha+\beta-\gamma}{\alpha}>1$ implying that both steady states $x=1$ and $x=0$ are locally unstable. Evaluating (18) at s^* gives

$$f'\left(s^*;b,N\right)=1+\frac{\left(1-s^*\right)s^*\beta\delta^*}{\alpha}=1+\psi$$

where $\psi\equiv\left(1-s^*\right)s^*\delta^*\beta/\alpha$. Notice that the negativity of δ^* is always negative (see proof of Lemma 1) implies that ψ is also negative and therefore $f'\left(s^*;b,N\right)<1$. The steady state $x=s^*$ is therefore stable if and only if $\psi>-2$. □

Proof of Proposition 4 First observe that for any value of α, any $\gamma > \beta$ and any $b \in (0,1)$ the derivative of $f(x;b,N)$, as given in (18), goes to $-\infty$ as $N \to \infty$. The latter is true since (18) is proportional to δ^* and from Lemma 1 it follows that $\delta^* \to -\infty$ as $N \to +\infty$. When $\beta = 2\gamma$ and $b = \frac{1}{2}$, we have $s^* = \frac{1}{2}$ (Proposition 1) for every N. As a result (18) can be written as

$$f'(s^*;b,N) = 1 + \frac{\delta_N^* \beta}{4\alpha}$$

where

$$\delta_N^* = -\frac{\left(\frac{1}{2}\right)^{N-2}(N-1)!}{\left(\frac{1}{2}N-1\right)!\left(\frac{1}{2}N-1\right)!}. \tag{19}$$

Notice that $\delta_2^* = 1$ and that for $N \geq 4$ we have $\delta_N^* = -\frac{3}{2} \times \frac{5}{4} \times \cdots \times \frac{N-1}{N-2}$. Notice that δ_N^* is monotonically decreasing in N and that $\delta_N^* \to -\infty$ as $N \to \infty$ (by Lemma 1). We then have that $\psi = \psi_N = -\frac{\beta}{4\alpha}\delta_N^*$. Thus there exists an integer M such that $\psi_N < -2$ when $N > M$. Since this is the local stability condition of s^*, we have proved that there exists an integer M such that s^* is locally stable if and only if $N \leq M$. □

Proof of Proposition 5 Recall that a function $f(x)$ is *even* when $f(-x) = f(x)$ for all x and *odd* when $f(-x) = -f(x)$ for all x. Rewriting (10) in terms of z gives

$$\Delta z_{t+1} = m(z_t) = \frac{n(z_t)}{d(z_t)} = \frac{\beta\left(p(z_t + \frac{1}{2}; \frac{N}{2}, N) - \frac{1}{2}\right)\left(z_t + \frac{1}{2}\right)\left(\frac{1}{2} - z_t\right)}{\left(z_t + \frac{1}{2}\right)\beta\left(p\left(z_t + \frac{1}{2}; \frac{N}{2}, N\right) - \frac{1}{2}\right) + \alpha}.$$

From (3) it follows that $p\left(z + \frac{1}{2}; \frac{N}{2}, N\right) + p\left(-z + \frac{1}{2}; \frac{N}{2}, N\right) = 1$ or $p\left(z + \frac{1}{2}; \frac{N}{2}, N\right) - \frac{1}{2} = -\left(p\left(-z + \frac{1}{2}; \frac{N}{2}, N\right) - \frac{1}{2}\right)$. Together with the fact that $\left(z + \frac{1}{2}\right)\left(\frac{1}{2} - z\right)$ is an even function of z this implies that $n(z)$ is odd, that is, $n(-z) = -n(z)$. Now consider $z \in \left(-\frac{1}{2}, 0\right)$. We then have

$$d(z) = \beta\left(z + \frac{1}{2}\right)\left(p\left(z + \frac{1}{2}; \frac{N}{2}, N\right) - \frac{1}{2}\right) + \alpha > 0$$

since $p\left(z + \frac{1}{2}; \frac{N}{2}, N\right) - \frac{1}{2}$ is positive if and only if z is negative. Moreover, we then also have

$$d(-z) = -\beta\left(\frac{1}{2} - z\right)\left(p\left(z + \frac{1}{2}; \frac{N}{2}, N\right) - \frac{1}{2}\right) + \alpha > 0,$$

where the inequality follows from the fact that the $d(-z)$ is bounded from below by $-\frac{\beta}{4} + \alpha$, which is positive since $\alpha - \gamma > 0$ and $\beta = 2\gamma$. So for $z \in \left(-\frac{1}{2}, 0\right)$ we have

$$d(z) - d(-z) = \beta\left(p\left(z + \frac{1}{2}; \frac{N}{2}, N\right) - \frac{1}{2}\right) > 0.$$

Summarizing, for all $z \in \left(-\frac{1}{2}, 0\right)$ we have $d(z) > d(-z) > 0$ and $-n(-z) = n(z) > 0$ which implies $-m(-z) > m(z)$. Similarly, for $z \in \left(0, \frac{1}{2}\right)$ it can be shown that $-m(z) > m(-z)$. Combining those we find that $m(z) + m(-z) < 0$, for $z \notin \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$.

For the route choice problem straightforward computations from the equivalent of (10) lead to:

$$\Delta z_{t+1} = r\left(z_t\right) = \frac{2\beta\left(p\left(z_t + \frac{1}{2}; \frac{N}{2}, N\right) - \frac{1}{2}\right)\left(z_t + \frac{1}{2}\right)\left(\frac{1}{2} - z_t\right)}{2z_t\beta\left(p\left(z_t + \frac{1}{2}; \frac{N}{2}, N\right) - \frac{1}{2}\right) + \alpha}.$$

Here the denominator is an even function and the numerator is odd, making $r(z)$ an odd function. □

Proof of Proposition 6 The steady states of (16) solve $x = \frac{xF}{xF+(1-x)\alpha}$ and $F = \alpha - \gamma + \beta p(x; N_c, N)$. The three steady states (x, F) are given by $(0, \alpha + \beta - \gamma)$, $(1, \alpha - \gamma)$ and (s^*, α), respectively. The Jacobian of (16), $J(x, \pi_1)$, is given by

$$\mathbf{J}|_{(x,F)} = \begin{pmatrix} \frac{\alpha F}{\overline{F}^2} & \frac{x(1-x)\alpha}{\overline{F}^2} \\ (1-\mu)\beta\delta\frac{\alpha F}{\overline{F}^2} & \mu + (1-\mu)\beta\delta\frac{x(1-x)\alpha}{\overline{F}^2} \end{pmatrix},$$

where $\overline{F} = xF + (1-x)\alpha$. It follows immediately that $J(0, \alpha - \gamma + \beta)$ has eigenvalues $\lambda_1 = \frac{F}{\alpha} = \frac{\alpha+\beta-\gamma}{\alpha} > 1$ and $\lambda_2 = \mu \in (0,1)$ and that $J(1, \alpha - \gamma)$ has eigenvalues $\lambda_1 = \frac{\alpha}{F} = \frac{\alpha}{\alpha-\gamma} > 1$ and $\lambda_2 = \mu \in (0,1)$. These boundary steady states are therefore unstable for any value of μ. The characteristic equation for the Jacobian evaluated at (s^*, α) is

$$\lambda^2 - [1 + \mu + (1-\mu)\psi]\lambda + \mu = 0 \tag{20}$$

where, as before, $\psi \equiv s^*(1-s^*)\frac{\delta^*\beta}{\alpha}$. When $\mu \in (0,1)$ one can show that the eigenvalues are complex as long as $-\frac{\left(1+\sqrt{\mu}\right)^2}{1-\mu} < \psi < -\frac{\left(1-\sqrt{\mu}\right)^2}{1-\mu}$. If this condition holds then we have$|\lambda_1| = |\lambda_2| = \mu \in (0,1)$ and the interior steady state is locally stable. If the eigenvalues are real the local stability conditions are given by $\lambda_1 < 1$ and $\lambda_2 > -1$ (where we have labeled the eigenvalues such that $\lambda_1 > \lambda_2$). From (20) it follows that $\psi(1-\mu) < 0$ implies that $\lambda_1 < 1$ always holds. On the other hand, $\lambda_2 > -1$ as long as $\psi > \psi^* = -2\frac{1+\mu}{1-\mu}$. Moreover, since for $\psi \leq \psi^*$ the eigenvalues are real it follows that the interior steady state (s^*, α) is locallys stable if and only if $\psi > \psi^*$. □

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